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Stochastic resonance in two-dimensional arrays of coupled nonlinear oscillators

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In this Brief Report we report the results of computer simulations on the periodic and noise driving of two-dimensional square arrays of coupled nonlinear oscillators. We find significant improvement in the output of these arrays over their one-dimensional counterparts (quantified by signal-to-noise ratio in the power spectrum at the frequency of the periodic driving). We also find that, within the limited resolution of our simulations, the one-dimensional scaling laws proposed by Lindner *et al.* [Phys. Rev. **E** 53, 2081 (1996)] seem to hold quite well for two-dimensional arrays.

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Over the last 20 years there has been much interest in the role of noise when bistable oscillators are driven with a subthreshold periodic signal [1,2]. The key observation is that the noise, at a certain optimum level, aids the transition of the system between its stable states at a frequency equal to the frequency of the periodic forcing. This is referred to as stochastic resonance and an exhaustive discussion may be found in the above references. A more recent interest is what happens when individual oscillators are coupled together so that the response of one oscillator has a direct impact on the behavior of its neighbors. This was addressed by Lindner *et al.* in 1995 and 1996 [3,4] in an investigation of one-dimensional chains of such oscillators. It was discovered that the response of the system [measured as the signal-to-noise ratio (SNR) of one of the oscillators] could be increased over that of a single oscillator. Scaling laws were also deduced for the variation of the SNR as a function of the number of oscillators, noise strength, and coupling strength.

In this paper we carry the work further by coupling the oscillators in two-dimensional arrays and investigating the role of array size and the effects of the coupling strengths between the oscillators. Although two-dimensional arrays were considered in [5] in the context of noise-enhanced propagation, no systematic investigation was made of the role of array size and the effects of the coupling strengths between the oscillators. We find that not only is the SNR significantly increased in a two-dimensional array but also the coupling strength required for best SNR is much smaller than for one-dimensional arrays.

The system studied is comprised of a two-dimensional square array of coupled, overdamped, nonlinear oscillators. The array is on a Cartesian grid and each oscillator is coupled to its four nearest neighbors. The equation of motion for the (m,n)th oscillator is given by

$$\frac{dx_{m,n}}{dt}=ax_{m,n}-bx_{m,n}^3+A\sin(\omega t)+\sigma N_{m,n}(t)+\epsilon(x_{m-1,n}+x_{m+1,n}+x_{m,n+1}-4x_{m,n}), \qquad (1)$$

where a and b are constants and A and ω are the amplitude and angular frequency, respectively, of the (subthreshold) periodic forcing. N is a zero-mean, unit-variance, Gaussian noise process with the noise being local to each element and uncorrelated with the noise at the other elements. The added noise is characterized by its variance σ^2, and is expressed in dB such that the noise level is $10\log_{10}(\sigma^2)$. ϵ represents the strength of coupling between the elements. We used free boundary conditions at the edges of the array.

We chose to use the parameters $a=2.1078$, $b=1.4706$, $A=1.3039$, and $f=\omega/2\pi=0.116$, which were introduced in [3], so it would be possible to make direct comparison with the earlier work. The time evolution of the system was obtained by numerically integrating Eq. (1) over 32 cycles of the periodic forcing and using 4096 time steps per cycle. The results presented here represent hundreds of hours of simulation on a 450 MHz Apple G4 computer.

The output of the system was taken from the central element of the array and the raw signal thresholded to eliminate the effects of intrawell motion. A small hysteresis was also applied to eliminate minor excursions across the barrier between the two wells. To assess the switching dynamics of the system we computed the SNR of the thresholded output signal. To calculate the SNR we subtracted the background noise at the forcing frequency in the power spectral density and divided this by the local background power. The SNR is

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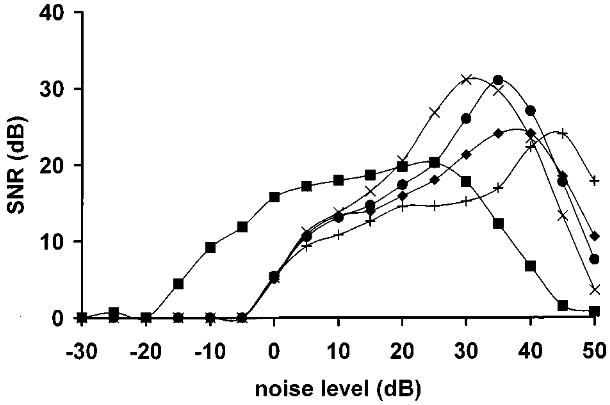

FIG. 1. SNR vs noise level for 121 oscillators arranged in an 11×11 grid for coupling strengths $\epsilon=0.1$ (■), $\epsilon=5.0$ (×), $\epsilon=10$ (●), and $\epsilon=100$ (+). For comparison, the response of a 121-element linear array is also plotted (♦) with a coupling constant $\epsilon=170$, which yields the best SNR for this size of array. The lines between data points are to guide the eye and the data points have a standard deviation of ± 0.5 dB.

expressed in decibels and, for our simulations, no windowing or other preprocessing of the signal was used. To obtain uncertainties on the SNR results we repeated the simulations four times.

Simulations were run to explore the effect of array size and coupling strengths. An example of the SNR vs noise curves for various coupling strengths is shown in Fig. 1. For comparison we also plot the response of the central element of a 121-element, one-dimensional array with its optimal coupling strength of $\epsilon=170$. It is clear that when the oscillators are arranged in a two-dimensional array the SNR is significantly larger than in the one-dimensional case.

We varied the coupling strength to find the best SNR for different sized arrays of oscillators and in Fig. 2 plot the best SNR obtained as a function of the number of oscillators. For comparison we have superimposed the SNR curve for one-dimensional arrays. The two-dimensional arrays appear to asymptotically approach ~33 dB, which is about 8 dB greater than the best SNRs that can be obtained using one-dimensional arrays.

Since the enhancement of the SNR must be due to the communication of excitation or switching across a number of

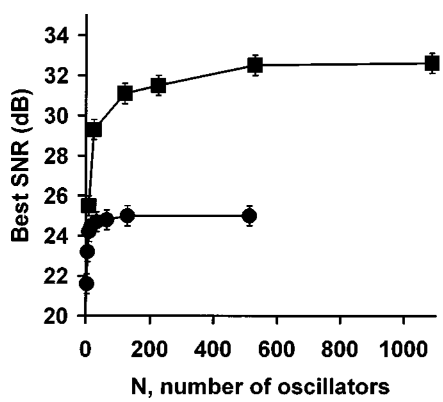

FIG. 2. Best SNR vs N, the total number of elements in the array. ●, one-dimensional arrays (data taken from Ref. [4]). ■, two-dimensional arrays of $N^{1/2}\times N^{1/2}$ elements. The best SNR for a single (zero-dimensional) oscillator is ~17 dB.

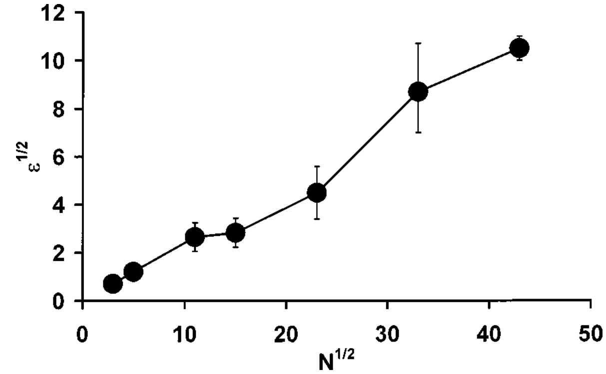

FIG. 3. Square root of the coupling constant at best SNR plotted against $N^{1/2}$, where N is the total number of elements in the two-dimensional arrays. The ''error bars'' are our estimates for the uncertainty in being able to locate the position of the optimal coupling and roughly show the breadth of the peak within 1 dB of the best SNR.

elements, it is interesting to examine the role of the coupling strength. It has been argued [4] that, for a one-dimensional array, the square root of the optimal coupling scales as the number of elements in the array. This argument first notes that when best SNR is reached the elements of the array are synchronized, switching in unison with the periodic forcing. Thus, excitations are spreading across the whole length of the array. It is then further recognized that the coupling strength determines the ''stiffness'' of the array and that the speed of propagation of a disturbance across such a system is proportional to the square root of the stiffness. Then, since the size of the one-dimensional arrays is proportional to the number of elements, it follows that the square root of the coupling should increase with the number of elements for best SNR.

Because the maximum linear dimension in the two-dimensional arrays grows as the square root of the number of elements, we would expect that the square root of the optimal coupling strength would increase as the square root of the number of elements. We determined the coupling strengths that led to the best SNR and Fig. 3 shows that the above scaling law does seem to be followed. An interesting implication of the above results is that the value of the coupling strength needed to obtain the best SNR is much smaller (by a factor of the square root of the number of elements) for a two-dimensional array of N oscillators than for a one-dimensional arrangement. It can also be noted that for larger arrays the exact value of the coupling strength becomes progressively less important. In fact, for our 33×33 and 43×43 arrays, the coupling can be varied over an order of magnitude with only ~2 dB variation in the SNR. This widening of the range of optimal coupling as N gets larger is also seen in one-dimensional arrays.

We also found that the optimal noise level, at the optimal coupling, for a given number of oscillators is approximately the same as in the one-dimensional case. This is illustrated in Fig. 1 for the case of 121-element arrays.

In conclusion, our results show clear similarities with the one-dimensional arrays of [3,4], and some quite striking differences. First, the best SNR achievable with two-dimensional arrays is some 8 dB greater than for one-dimensional arrays. One may suppose that this occurs because the excitation that spreads out from a switched oscillator can communicate with other oscillators by a number of routes. Thus switching, when it is supposed to occur during one of the half periods of the periodic forcing, is not blocked by one particularly recalcitrant element as it could be in the one-dimensional array.

A second feature is the considerably lower coupling strengths that are needed to obtain best SNR compared to one-dimensional arrays. This is reasonable from a dimensional standpoint if we follow the argument of [4] and associate the square root of the coupling strength with the velocity of propagation of the excitation. This might be of interest for practical applications of stochastic resonance if only small coupling strengths were available or preferred.

This work has been carried out in support of an optical experimental system that we are currently building to study array-enhanced stochastic resonance. Results from these experiments and further refinements of our computer model (including, for example, distributions of coupling strengths) will be reported later.

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